NILPOTENT GROUP C*-ALGEBRAS AS COMPACT QUANTUM METRIC SPACES

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Abstract. Let $\mathbb{L}$ be a length function on a group G, and let $M_{\mathbb{L}}$ denote the operator of pointwise multiplication by $\mathbb{L}$ on $\ell^2(G)$. Following Connes, $M_{\mathbb{L}}$ can be used as a "Dirac" operator for the reduced group C*-algebra $C_r^*(G)$. It defines a Lipschitz seminorm on $C_r^*(G)$, which defines a metric on the state space of $C_r^*(G)$. We show that for any length function of a strong form of polynomial growth on a discrete group, the topology from this metric coincides with the weak-$*$ topology (a key property for the definition of a "compact quantum metric space"). In particular, this holds for all word-length functions on finitely generated nilpotent-by-finite groups.

1. Introduction

The group C^*-algebras of discrete groups provide a much-studied class of "compact non-commutative spaces" (that is, unital C^*-algebras). In [5] Connes showed that the "Dirac" operator of a spectral triple over a unital C^*-algebra provides in a natural way a metric on the state space of the algebra. The class of examples most discussed in [5] consists of the group C^*-algebras of discrete groups G, with the Dirac operator consisting of the pointwise multiplication operator on $\ell^2(G)$ by a word-length function on the group. In [17, 18] the second author pointed out that, motivated by what happens for ordinary compact metric spaces, it is natural to desire that a spectral triple have the property that the topology from the metric on the state space coincide with the weak-$*$ topology (for which the state space is compact). This property was verified in [17] for certain examples. In [20] this property was taken as the key property for the definition of a "compact quantum

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metric space". This property is crucial for defining effective notions of quantum Gromov-Hausdorff distance between compact quantum metric spaces [20, 22, 23, 24].

In [19] the second author studied this property for Connes' original class of examples consisting of discrete groups with Dirac operators coming from a word-length functions, and established that it holds for the group $\mathbb{Z}^n$, relying on geometric arguments. Later, with N. Ozawa [12], he established this property for hyperbolic groups with word-length functions. The argument was very different from that in [19], relying on filtered C*-algebras.

In the present paper we verify the property for the case of finitely generated nilpotent-by-finite groups equipped with length functions of polynomial growth, and generalize this to a certain class of length functions on infinitely generated discrete groups. Since the approach used in the present paper is quite different from those used in [19] and [12], this raises the question of finding a unified approach which covers both the nilpotent and hyperbolic settings. The question of what happens for other classes of groups remains wide open.

To be more specific, let G be a countable (discrete) group, and let $c_c = C_c(G)$ denote the convolution $*$-algebra of complex-valued functions of finite support on G. Let λ denote the usual $*$-representation of c_c on $\ell^2 = \ell^2(G)$ coming from the unitary representation of G by left translation on ℓ^2. Thus

$$\lambda_f(\xi)(x) = f * \xi(x) = \sum_{y \in G} f(xy^{-1})\xi(y)$$

for functions $\xi \in \ell^2(G)$. The completion of $\lambda(c_c)$ for the operator norm is by definition the reduced group C^*-algebra, $C_r^*(G)$, of G. We identify c_c with its image in $C_r^*(G)$, so that it is a dense $*$-subalgebra. We remark that by sending an element $a \in C_r^*(G)$ to the element of ℓ^2 to which it sends $\delta_e \in \ell^2$ we obtain an embedding of $C_r^*(G)$ into ℓ^2. Thus when convenient we can view all of the elements of $C_r^*(G)$ as functions on G. We denote by e the identity element of G.

The Følner condition for amenability [11, 15] is a simple consequence of polynomial growth (in the weakest of the three versions defined below). Consequently the full and reduced group C*-algebras coincide [15] under our hypotheses, and so we do not need to distinguish between them.

Let a length function $\mathbb{L}$ be given on G. That is, $\mathbb{L}$ is a function from G to $[0, \infty)$ that satisfies

(1) $\mathbb{L}(xy) \leq \mathbb{L}(x) + \mathbb{L}(y)$ for all $x, y \in G$;
(2) $\mathbb{L}(x^{-1}) = \mathbb{L}(x)$ for all $x \in G$;

(3) $\mathbb{L}(x) = 0$ if and only if $x = e$.

We say that $\mathbb{L}$ is proper if $B(r) = \{x \in G : \mathbb{L}(x) \leq r\}$ is a finite subset of G for each $r < \infty$.

Throughout the paper, we denote by $|E|$ the cardinality of a finite set E.

In the literature there are actually two (or more) inequivalent definitions of "polynomial growth". Since we want to distinguish between them, we will call one of them "strong polynomial growth". The proof of our main theorem works most naturally for an intermediate property, which we call "bounded doubling".

Definition 1.1. Let $\mathbb{L}$ be a length function on a group G. We say that $\mathbb{L}$ has (or is of)

(1) *strong polynomial growth* if $\mathbb{L}$ is proper and there exist constants $C_{\mathbb{L}} < \infty$ and $d < \infty$ such that

$$C_{\mathbb{L}}^{-1} r^d \leq |B(r)| \leq C_{\mathbb{L}} r^d \quad \text{for all } r \geq 1. \tag{1.1}$$

(2) *bounded doubling* if $\mathbb{L}$ is proper and there exists a constant $C_{\mathbb{L}} < \infty$ such that

$$|B(2r)| \leq C_{\mathbb{L}} |B(r)| \quad \text{for all } r \geq 1. \tag{1.2}$$

(3) *polynomial growth* if $\mathbb{L}$ is proper and there exist constants $C_{\mathbb{L}} < \infty$ and $d < \infty$ such that

$$|B(r)| \leq C_{\mathbb{L}} r^d \quad \text{for all } r \geq 1. \tag{1.3}$$

Equivalent definitions are obtained by changing the restriction $r \geq 1$ to $r \geq r_0$ for any $r_0 > 0$, but the constants $C_{\mathbb{L}}$ may depend on r_0.

Proposition 1.2. *Let $\mathbb{L}$ be a length function on a group G. If $\mathbb{L}$ has strong polynomial growth, then it has bounded doubling. If $\mathbb{L}$ has bounded doubling then it has polynomial growth. If G is finitely generated, then these three properties are equivalent. But in general, no two of these properties are equivalent.*

See Section 5 for a proof, and for examples illustrating these distinctions.

We let M_h denote the (often unbounded) operator on ℓ^2 of pointwise multiplication by a function $h : G \to \mathbb{C}$. The multiplication operator $M_{\mathbb{L}}$ will serve as our "Dirac" operator, and we will denote it by D. One sees easily [6, 19, 12] that the commutators $[D, \lambda_f]$ are bounded operators for each $f \in c_c$. We can thus define a seminorm, L_D, on c_c by $L_D(f) = \|[D, \lambda_f]\|$, where $\|T\|$ denotes the operator norm of a bounded linear operator $T : \ell^2(G) \to \ell^2(G)$. (Connes points out in proposition 6 of [6] that $\mathbb{L}$ has polynomial growth exactly if there is

a positive constant, p, such that the operator $D = M_{\mathbb{L}}$ is such that $(1 + D^2)^{-p}$ is a trace-class operator.)

Let L be a $*$-seminorm (i.e. $L(a^*) = L(a)$) on a dense $*$-subalgebra A of a unital C^*-algebra $\bar{A}$, satisfying $L(1) = 0$. Define a metric, ρ_L, on the state space $S(\bar{A})$ of $\bar{A}$, much as Connes did, by

$$\rho_L(\mu, \nu) = \sup\{|\mu(a) - \nu(a)| : a \in A,\ L(a) \leq 1\}.$$

(Without further hypotheses, ρ_L may take the value $+\infty$.)

Definition 1.3. [18] A $*$-seminorm L on A is a *Lip-norm* if the topology on $S(\bar{A})$ defined by the associated metric ρ_L coincides with the weak-$*$ topology.

We consider a unital C^*-algebra equipped with a Lip-norm L to be a compact quantum metric space [20], but for many purposes one wants L to satisfy further properties. See the discussion after Proposition 1.6. The main question that we deal with in this paper is whether the seminorms L_D defined as above in terms of length functions $\mathbb{L}$ on discrete groups are Lip-norms. Our main theorem is:

Theorem 1.4. *Let G be a discrete group, and let $\mathbb{L} : G \to [0, \infty)$ be a length function of bounded doubling on G. Let $D = M_{\mathbb{L}}$ be the associated multiplication operator. Then the seminorm L_D defined on c_c by $L_D(f) = \|[D, \lambda_f]\|$ is a Lip-norm on $C^*(G)$.*

Necessary and sufficient conditions for a seminorm on a pre-C^*-algebra to be a Lip-norm are given in [17, 18] (in a more general context). For our present purposes it is convenient to reformulate these conditions slightly. The following reformulation is an immediate corollary of proposition 1.3 of [12].

Proposition 1.5. *Let G be a discrete group, and let $\mathbb{L} : G \to [0, \infty)$ be a length function. The associated seminorm L_D is a Lip-norm on $c_c = C_c(G)$ if and only if λ carries*

$$\{f \in c_c : f(e) = 0 \text{ and } L_D(f) \leq 1\}$$

to a subset of $\mathcal{B}(\ell^2)$ that is totally bounded for the operator norm.

Accordingly, the content of this paper consists in verifying the criterion of this proposition for the case of a group G equipped with a length function $\mathbb{L}$ that has bounded doubling.

Shorn of its functional analytic context and motivation, the result proved in this paper is as follows. The proof developed below is loosely related to some elements of [4] and [3].

Proposition 1.6. *Let G be a discrete group. Let $\mathbb{L} : G \to [0, \infty)$ be a length function on G that has bounded doubling, and let $D_{\mathbb{L}}$ be the associated Dirac operator on $c_c(G)$. For every $\varepsilon > 0$ there exists a finite set $S_\varepsilon \subset G$ such that for any finitely supported $f : G \to \mathbb{C}$ satisfying $\|[D_{\mathbb{L}}, \lambda_f]\| \leq 1$ there exists a decomposition $f = f_\sharp + f_\flat$ such that $f_\flat$ is supported on S_ε and $\|\lambda_{f_\sharp}\| \leq \varepsilon$.*

More generally, for an arbitrary function $f : G \to \mathbb{C}$, $[D_{\mathbb{L}}, \lambda_f]$ is well-defined as a linear operator from c_c to the space of all functions from G to $\mathbb{C}$. The analysis below demonstrates that if $f : G \to \mathbb{C}$ is any function for which $[D_{\mathbb{L}}, \lambda_f]$ maps c_c to ℓ^2 and extends to a bounded linear operator from ℓ^2 to ℓ^2 with $\|[D_{\mathbb{L}}, \lambda_f]\| \leq 1$, then f satisfies the conclusion of Proposition 1.6. In particular, f (that is, λ_f) is necessarily an element of $C_r^*(G)$.

We believe that our whole discussion could be extended to the slightly more general setting of group C^*-algebras twisted by a 2-cocycle [13, 14], much as done in [19], but we have not checked this carefully.

The definition of a "compact C*-metric" as given in definition 4.1 of [23] brings together most of the additional conditions that have been found to be useful to require of a Lip-norm L on a C*-normed algebra $\mathcal{A}$. Namely, one wants L to be lower semi-continuous with respect to the operator norm, to be strongly Leibniz as defined there, and one wants the $*$-subalgebra of elements of $\mathcal{A}$ on which L is finite to be a dense spectrally-stable subalgebra of the norm-completion $\bar{\mathcal{A}}$ of $\mathcal{A}$. For any group G with proper length function $\mathbb{L}$ and corresponding seminorm L_D for $D = M_{\mathbb{L}}$ one can always obtain these properties in the following way (as explained in [23], especially its example 4.4). The one-parameter unitary group generated by D consists of the operators of pointwise multiplication by the functions $e^{it\mathbb{L}}$. Conjugation by these operators defines a one-parameter group, α, of automorphisms of $\mathcal{B}(\ell^2)$ (which need not be strongly continuous, and need not carry $\mathcal{A} = C_r^*(G)$ into itself). By using α one shows that L_D on c_c is lower semi-continuous with respect to the operator norm, and so has a natural extension, $\bar{L}_D$ to a lower semi-continuous seminorm on all of $\bar{\mathcal{A}} = C_r^*(G)$ (which may take the value $+\infty$). Let $\mathcal{A}^\infty$ denote the $*$-subalgebra of elements of $\bar{\mathcal{A}}$ that are infinitely differentiable for α It contains c_c and so is dense in $\bar{\mathcal{A}}$, and it is spectrally stable in $\bar{\mathcal{A}}$. The restriction of $\bar{L}_D$ to $\mathcal{A}^\infty$ satisfies all the conditions for being a C^*-metric, for reasons given in section 3 of [23], except for the fact that it may not be a Lip-norm. Thus this paper verifies, for groups with length functions of bounded doubling, the most difficult condition, namely of obtaining a Lip-norm, so that for such groups $(\mathcal{A}^\infty, \bar{L}_D)$ is a compact C*-metric

space. One can continue to show that all continues to work well for matrix algebras over $\mathcal{A}$ along the lines given in [24], so that one should give the definition of a "matricial C^*-metric", but we will not pursue that important aspect here.

Since both nilpotent-by-finite groups and hyperbolic groups are groups of "rapid decrease" [9, 7], it is natural to ask whether our main theorem extends to all groups of rapid decrease. For the reader's convenience we recall here the definition of this concept: For any group G and length function $\mathbb{L}$ on it, and for any $s \in \mathbb{R}$, the Sobolev space $\mathcal{H}^s_{\mathbb{L}}(G)$ is defined to be the set of functions ξ on G such that $(1+\mathbb{L})^s\xi \in \ell^2$. The space $\mathcal{H}^\infty_{\mathbb{L}}$ of rapidly decreasing functions is defined to be $\bigcap_{s\in\mathbb{R}} \mathcal{H}^s_{\mathbb{L}}$. The group G is said to be of rapid decrease if it has a length function $\mathbb{L}$ such that $\mathcal{H}^\infty_{\mathbb{L}}$ is contained in $C^*_r(G)$, that is, if all the convolutions of elements of c_c by elements of $\mathcal{H}^\infty_{\mathbb{L}}$ extend to bounded operators on ℓ^2. For closely related Lip-norms (which are not Leibniz) obtained by using "higher derivatives" for groups of rapid decrease, see [1].

2. Localized weighted inequality

In this section we develop a key inequality that holds for any discrete group G equipped with a proper length function $\mathbb{L}$. For any $h \in \ell^\infty$ we let M_h denote the operator on ℓ^2 of pointwise multiplication by h. If E is a subset of G, we let M_E denote M_h for h the characteristic (or indicator) function χ_E of E, so $M_E = M_{\chi_E}$ is a projection operator. For any $r \geq 0$ we set $B(r) = \{x \in G : \mathbb{L}(x) \leq r\}$, which is a finite set since $\mathbb{L}$ is proper. We set $M_r = M_{B(r)}$. Each M_r is a spectral projection of D.

It is convenient to use the kernel functions for the operators λ_f and $[D, \lambda_f]$, for any $f \in c_c$. The kernel function for λ_f is $f(xy^{-1})$, that is, $(\lambda_f\xi)(x) = \sum_y f(xy^{-1})\xi(y)$ for any $\xi \in \ell^2$. The kernel function $[D, \lambda_f](x, y)$ for the operator $[D, \lambda_f]$ is $[D, \lambda_f](x, y) = (\mathbb{L}(x) - \mathbb{L}(y))f(xy^{-1})$, with slight abuse of notation. Thus if $\mathbb{L}(x) \neq \mathbb{L}(y)$ then

$$f(xy^{-1}) = (\mathbb{L}(x) - \mathbb{L}(y))^{-1}[D, \lambda_f](x, y).$$

If $\mathbb{L}(x) > \mathbb{L}(y)$ then

$$(\mathbb{L}(x) - \mathbb{L}(y))^{-1} = \mathbb{L}(x)^{-1}(1 - \mathbb{L}(y)/\mathbb{L}(x)) = \mathbb{L}(x)^{-1}\sum_{k=0}^{\infty}\mathbb{L}(y)^k\mathbb{L}(x)^{-k}.$$

Thus, if we are given $r, s \in [0, \infty)$ with $0 \leq r < s$, and if $\xi \in \ell^2$ is supported in $B(r)$, then for any $x \in G$ satisfying $\mathbb{L}(x) \geq s$ we have

$$(\lambda_f\xi)(x) = \sum_y f(xy^{-1})\xi(y) = \sum_y (\mathbb{L}(x) - \mathbb{L}(y))^{-1}[D, \lambda_f](x, y)\xi(y)$$

$$
\begin{aligned}
&= \sum_{y\in B(s)} \mathbb{L}(x)^{-1} \sum_k \mathbb{L}(x)^{-k}\mathbb{L}(y)^k [D,\lambda_f](x,y)\xi(y) \\
&= \sum_k \mathbb{L}(x)^{-1} \sum_{y\in B(s)} \mathbb{L}(x)^{-k}[D,\lambda_f](x,y)\mathbb{L}(y)^k\xi(y) \\
&= \big(\sum_k D^{-1-k}(I-M_s)[D,\lambda_f]D^k M_r\xi\big)\ (x).
\end{aligned}
$$

That is,

$$(I-M_s)\lambda_f M_r = \sum_{k=0}^{\infty} D^{-1-k}(I-M_s)[D,\lambda_f]D^k M_r.$$

But $\|D^{-1-k}(I-M_s)\| \leq s^{-1-k}$ while $\|[D,\lambda_f]D^k M_r\| \leq r^k L_D(f)$. Consequently

$$\|(I-M_s)\lambda_f M_r\| \ \leq \ s^{-1}\sum_k (r/s)^k L_D(f) = (s-r)^{-1}L_D(f).$$

We have thus obtained:

Proposition 2.1. *For any $f \in c_c$ and any $r, s \in \mathbb{R}$ with $s > r \geq 0$ we have*

$$\|(I-M_s)\lambda_f M_r\| \ \leq \ (s-r)^{-1}L_D(f).$$

Let us compare this proposition with the main result of section 2 of [12]. Suppose that $\mathbb{L}$ takes its values in $\mathbb{N}$, and for each $n \in \mathbb{N}$ let $\mathcal{A}_n$ consist of the elements of c_c supported on $B(n)$. Let $\mathcal{A}$ denote the union of the $\mathcal{A}_n$'s, so that $\mathcal{A}$ is a unital dense $*$-subalgebra of $\ell^1(G)$. Then the family $\{\mathcal{A}_n\}$ is a filtration of $\mathcal{A}$, and in the topological sense it is a filtration of $\ell^1(G)$, and of the C*-algebra completion $C_r^*(G)$ of $\ell^1(G)$ for the operator norm. This is discussed in section 1 of [12], where the following observations are made. For a faithful tracial state on a filtered C*-algebra (such as the canonical trace on $C_r^*(G)$) with filtration $\{\mathcal{A}_n\}$, one can form the corresponding GNS Hilbert space, $\mathcal{H}$, and the representation λ of $\mathcal{A}$ on it coming from the left regular representation of $\mathcal{A}$ on itself. For each $n \in \mathbb{N}$ let Q_n denote the orthogonal projection of $\mathcal{H}$ onto its (finite-dimensional) subspace $\mathcal{A}_n$. (In the above discussion for groups this operator would be denoted by M_n.) Then set $P_n = Q_n - Q_{n-1}$ for $n \geq 1$, and $P_0 = Q_0$. The P_n's are mutually orthogonal, and their sum is $I_{\mathcal{H}}$ for the strong operator topology. One then defines an unbounded operator D on $\mathcal{H}$ by $D = \sum_{n=0}^{\infty} nP_n$. For any $a \in \mathcal{A}$ the densely defined operator $[D,\lambda_a]$ is a bounded operator, and so extends to a bounded operator on $\mathcal{H}$. We can then define

a seminorm, L_D, on $\mathcal{A}$ by

$$L_D(a) = \|[D, \lambda_a]\|.$$

This L_D is essentially a generalization of the L_D that we have used above for the group case. Let T be any bounded operator on $\mathcal{H}$ such that $[D, T]$ has dense domain containing $\mathcal{A}$ and is bounded on its domain, so extends to a bounded operator on $\mathcal{H}$. For any natural number N set

$$T^{(N)} = \sum_{|m-n|>N} P_m T P_n.$$

Then the main result of section 2 of [12] provides a specific sequence, $\{C_N\}$, of constants, independent of D and T, that converges to 0 as N goes to ∞, such that

$$\|T^{(N)}\| \leq C_N \|[D, T]\|$$

for all N. Notice then that for any $p, q \in \mathbb{N}$ such that $q - p > N$ we have

$$(1 - Q_q)\Big(\sum_{|m-n|>N} P_m T P_n \Big) Q_p = (1 - Q_q) T Q_p,$$

and consequently

$$\|(1 - Q_q) T Q_p)\| \leq C_N \|[D, T]\|. \tag{2.1}$$

This is essentially a generalization of Proposition 2.1, but with not as good a constant.

3. Cutoff functions

For the proof of Theorem 1.4 we seek, for any $\varepsilon > 0$ and every $f \in c_c$, a decomposition $f = f_\sharp + f_\flat$ with certain properties. It is natural to accomplish this by means of multiplication operators, so that in the notation of Proposition 1.6, $f_\flat = M_g f = gf$ where the cutoff function g depends only on G, $\mathbb{L}$, and ε. It will be more convenient to construct $f_\sharp$ in this way, and this will be accomplished by means of an infinite series of finitely supported cutoff functions. Thus one is led to analyze $\lambda_{g_\nu f}$ in terms of λ_f, for a family of cutoff functions g_ν whose supports are finite for each ν, but not uniformly so.

As motivation, consider the Abelian case, employing additive notation $x - y$ in place of multiplicative xy^{-1} for the group operation. The operator λ_{gf} has kernel function $g(x - y) f(x - y)$. As in the proof of Proposition 2.1, it can be useful to express g as an infinite sum of product functions $g(x - y) = \sum_k \phi_k(x) \psi_k(y)$ with $\sum_k \|\phi_k\|_{L^\infty} \|\psi_k\|_{L^\infty} \leq C_0$, where C_0 is a finite constant which is to be bounded uniformly over a suitable family of cutoff functions g. This expresses λ_{gf} as

$\sum_k M_{\phi_k} \lambda_f M_{\psi_k}$ with $\sum_k \|M_{\phi_k} \lambda_f M_{\psi_k}\| \le C_0 \|\lambda_f\|$. If the Fourier transform $\widehat{g}$ satisfies $\|\widehat{g}\|_{L^1} \le C_0$ then one obtains at once a continuum decomposition of this type;

$$g(x-y) = \int \widehat{g}(\xi) e^{2\pi i \xi \cdot (x-y)} \, d\xi = \int \widehat{g}(\xi) e^{2\pi i \xi \cdot x} e^{-2\pi i \xi \cdot y} \, d\xi,$$

and one sets $\phi_\xi(x) = \widehat{g}(\xi) e^{2\pi i x \cdot \xi}$ and $\psi_\xi(y) = e^{-2\pi i y \cdot \xi}$ to obtain

$$\int \|\phi_\xi\|_\infty \|\psi_\xi\|_\infty \, d\xi \le C_0.$$

One effective way to ensure that $\|\widehat{g}\|_{L^1} \le C_0$ is to express g as a convolution product $g = g_1 * g_2$ with $\|g_1\|_{\ell^2} \|g_2\|_{\ell^2} \le C_0$. For not necessarily Abelian groups with length functions of bounded doubling, we will show below how convolution products of appropriately chosen ℓ^2 functions can be used to construct useful cutoff functions g, despite the lack of a convenient Fourier transform.

3.1. **Convolutions as cutoff functions.** We begin with some generalities concerning λ_{gf} when the cutoff function g is expressed as a convolution $h^* * k$ for $h, k \in c_c$. Let ρ denote the right regular representation of G on ℓ^2, defined by $\rho_u(\xi)(x) = \xi(xu^{-1})$. Then ρ_u commutes with λ_f for any $f \in c_c$. For any $h \in c_c$ we define $\tilde{h}(x) = h(x^{-1})$ and $h^*(x) = \overline{h}(x^{-1})$.

Proposition 3.1. *For any $f, h, k \in c_c$ we have*

$$\lambda_{(h^* * k)f} = \sum_z \rho_z^* M_{\tilde{h}}^* \lambda_f M_{\tilde{k}} \rho_z, \tag{3.1}$$

where this sum converges for the weak operator topology. Furthermore

$$\|\lambda_{(h^* * k)f}\| \le \|\lambda_f\| \|h\|_2 \|k\|_2.$$

Proof. Notice that

$$(h^* * k)(yx^{-1}) = \sum_z \bar{h}(z^{-1}) k(z^{-1} y x^{-1}) = \sum_z \bar{h}(z^{-1} y^{-1}) k(z^{-1} x^{-1}).$$

Then, on using this, for any $\xi, \eta \in c_c$ we have

$$\begin{aligned}
\langle \lambda_{(h^* * k)f} \xi, \eta \rangle &= \sum_y (\lambda_{(h^* * k)f} \xi)(y) \bar{\eta}(y) \\
&= \sum_y \sum_x (h^* * k)(yx^{-1}) f(yx^{-1}) \xi(x) \bar{\eta}(y) \\
&= \sum_y \sum_x \sum_z \bar{h}(z^{-1} y^{-1}) k(z^{-1} x^{-1}) f(yx^{-1})) \xi(x) \bar{\eta}(y)
\end{aligned}$$

$$= \sum_y \sum_x \sum_z f(yx^{-1})k(x^{-1})\xi(xz^{-1})\bar{h}(y^{-1})\bar{\eta}(yz^{-1})$$
$$= \sum_z \langle \lambda_f M_{\tilde{k}} \rho_z \xi,\ M_{\tilde{h}} \rho_z \eta \rangle$$
$$= \sum_z \langle \rho_z^* M_{\tilde{h}}^* \lambda_f M_{\tilde{k}} \rho_z \xi,\ \eta \rangle.$$

But

$$\sum_u \|M_{\tilde{k}} \rho_u \xi\|_2^2 = \sum_u \sum_x |M_{\tilde{k}} \rho_u \xi(x)|^2 = \sum_u \sum_x |k(x^{-1})\xi(xu^{-1})|^2$$
$$= \sum_x |k(x^{-1})|^2 \|\xi\|_2^2 = \|k\|_2^2 \|\xi\|_2^2,$$

and similarly for $M_{\tilde{h}} \rho_v \eta$, so that by Cauchy-Schwarz,

$$\sum_z |\langle \lambda_f M_{\tilde{k}} \rho_z \xi,\ M_{\tilde{h}} \rho_z \eta \rangle| \leq \|\lambda_f\| \|h\|_2 \|k\|_2 \|\xi\|_2 \|\eta\|_2.$$

This implies both convergence of the series (3.1) for the weak operator topology, and the stated norm inequality. Notice that because ρ is a unitary representation the norm of each operator $\rho_z^* M_{\tilde{h}}^* \lambda_f M_{\tilde{k}} \rho_z$ is equal to $\|M_{\tilde{h}}^* \lambda_f M_{\tilde{k}}\|$. □

Proposition 3.1 fits very well into the setting of "proper actions of groups on C*-algebras" that is defined and discussed in [16]. Let $\mathcal{A}$ denote the algebra of compact operators on ℓ^2, and let α denote the action of G on $\mathcal{A}$ by conjugation by ρ. From example 2.1 of [16] but with the roles of λ and ρ reversed, we see that α is a proper action as defined in [16]. The finite-rank operator $M_{\tilde{h}}^* \lambda_f M_{\tilde{k}}$ above is easily seen to have kernel function of finite support, putting it in the dense subalgebra $\mathcal{A}_0$ of example 2.1 of [16]. Accordingly $\sum_z \alpha_z(M_{\tilde{h}}^* \lambda_f M_{\tilde{k}})$ exists in the weak sense discussed in [16], and this sum is an element of the "generalized fixed-point algebra" for α as defined in [16]. Towards the end of example 2.1 it is explained that this generalized fixed-point algebra is, in the case of this example, just the C*-algebra generated by the left regular representation (for the roles reversed). Our proposition above yields $\lambda_{(h^* * k)f}$, which is indeed in this C*-algebra. This general setting is explored further in [21], especially in sections 7 and 8.

We do not, strictly speaking, need the following proposition, but it provides some perspective on the path that we will take below, e.g. in Proposition 4.6.

Proposition 3.2. *Let $f, h, k \in c_c$. Then*

$$L_D(\lambda_{(h^* * k)f}) \leq \|h\|_2 \|k\|_2 L_D(f).$$

Proof. Because $(h^* * k)f$ has finite support, $[D, \lambda_{(h^**k)f}]$ is a bounded operator. Let $\xi, \eta \in c_c$, so they are in the domain of D. Then

$$\langle [D, \lambda_{(h^**k)f}]\xi,\ \eta\rangle = \langle \lambda_{(h^**k)f}\xi,\ D\eta\rangle - \langle \lambda_{(h^**k)f}D\xi,\ \eta\rangle,$$

so by Proposition 3.1

$$\langle [D, \lambda_{(h^**k)f}]\xi,\ \eta\rangle = \sum_z \langle D\rho_z^* M_h^* \lambda_f M_k \rho_z \xi,\ \eta\rangle - \langle \rho_z^* M_h^* \lambda_f M_k \rho_z D\xi,\ \eta\rangle.$$

But, if by slight abuse of notation we let $\rho_z(h)$ denote the corresponding right translate of h, we see that $\rho_z^* M_h^* \rho_z = M_{\rho_z(h)}^*$, which commutes with D, and similarly for M_k. Furthermore ρ_z commutes with λ_f. It follows that

$$\langle [D, \lambda_{(h^**k)f}]\xi,\ \eta\rangle = \sum_z \langle \rho_z^* M_h^* [D, \lambda_f] M_k \rho_z \xi,\ \eta\rangle.$$

Consequently

$$|\langle [D, \lambda_{(h^**k)f}]\xi,\ \eta\rangle| \le L_D(f)\|h\|_2\|k\|_2\|\xi\|_2\|\eta\|_2$$

for much the same reasons as given near the end of the proof of Proposition 3.1. ∎

3.2. **The seminorm J_D and cutoff functions.** Later in the proof we will partly lose control of $L_D(gf)$ for certain functions g of interest. It is possible to retain some control, as follows. Notice that if, for any $r > 0$, we set $s = 2r$ in Proposition 2.1, we obtain

$$\|(I - M_{2r})\lambda_f M_r\| \ \le\ r^{-1}L_D(f).$$

This motivates the following definition.

Definition 3.3. The seminorm J_D on c_c is defined by

$$J_D(f) = \sup\{r\|(I - M_{2r})\lambda_f M_r\| : r > 0\}$$

for any $f \in c_c$.

The inequality

$$J_D(f) \le L_D(f) \ \text{ for all } f \in c_c$$

is an equivalent formulation of the special case $s = 2r$ of Proposition 2.1.

We emphasize that for the rest of this section, and for much of the next, we use J_D but not L_D, although some steps do have versions for L_D. Only near the end of the next section will we use the fact that $J_D \le L_D$. We will need:

Proposition 3.4. *Let $f \in c_c$. If $f(x) \neq 0$ for some $x \neq e$, then $J_D(f) \neq 0$. Thus the seminorm J_D is a norm on the subspace $\{f \in c_c : f(e) = 0\}$.*

Proof. Let δ_e be the delta-function at e, viewed as an element of ℓ^2. Then for any $r > 0$ we have $((I - M_{2r})\lambda_f M_r)(\delta_e) = (I - M_{2r})(f)$, where on the right-hand side f is viewed as an element of ℓ^2. Let $x \in G$ be such that $f(x) \neq 0$ and $x \neq e$ so that $\mathbb{L}(x) \neq 0$. Choose $r > 0$ such that $2r < \mathbb{L}(x)$. Then $(M_{2r}f)(x) = 0$, so that $(I - M_{2r})(f)(x) \neq 0$, and thus $J_D(f) \neq 0$. □

We now proceed to develop properties of J_D with respect to cutoffs of functions.

Proposition 3.5. *For a given $r > 0$, suppose that h is supported on $G \setminus B(2r)$ and that k is supported on $B(r)$. Then for any $f \in c_c$ we have*

$$\|\lambda_{(h^*k)f}\| \leq r^{-1}\|h\|_2\|k\|_2 J_D(f).$$

Proof. For any $\xi, \eta \in c_c$ we have, by Proposition 3.1,

$$\begin{aligned}
|\langle \lambda_{(h^* * k)f}\xi,\ \eta\rangle| &= |\sum_z \langle \lambda_f M_{\tilde{k}}\rho_z\xi,\ M_{\tilde{h}}\rho_z\eta\rangle| \\
&= |\sum_z \langle (I - M_{2r})\lambda_f M_r M_{\tilde{k}}\rho_z\xi,\ M_{\tilde{h}}\rho_z\eta\rangle| \\
&\leq r^{-1} J_D(f) \sum_z \|M_{\tilde{k}}\rho_z\xi\|_2 \|M_{\tilde{h}}\rho_z\eta\|_2 \\
&\leq r^{-1} (\sum_u \|M_{\tilde{k}}\rho_u\xi\|_2^2)^{1/2} (\sum_v \|M_{\tilde{h}}\rho_v\eta\|_2^2)^{1/2} J_D(f) \\
&= r^{-1}\|h\|_2\|k\|_2\|\xi\|_2\|\eta\|_2 J_D(f),
\end{aligned}$$

for reasons given near the end of the proof of Proposition 3.1. □

Quite parallel to Proposition 3.2 we have:

Proposition 3.6. *Let $f, h, k \in c_c$. Then*

$$J_D((h^* * k)f) \leq \|h\|_2\|k\|_2 J_D(f).$$

Proof. The justifications for the calculations in the proof are very similar to those in the proof of Proposition 3.2. For any $r > 0$ we have, by Proposition 3.1,

$$\begin{aligned}
&|\langle (I - M_{2r})\lambda_{(h^* * k)f} M_r\xi,\ \eta\rangle| = |\langle \lambda_{(h^* * k)f} M_r\xi,\ (I - M_{2r})\eta\rangle| \\
&= |\sum_z \langle \lambda_f M_{\tilde{k}}\rho_z M_r\xi,\ M_{\tilde{h}}\rho_z (I - M_{2r})\eta\rangle|
\end{aligned}$$

$$
\begin{aligned}
&= |\sum_z \langle (I - M_{2r})\lambda_f M_r M_{\tilde{k}} \rho_z \xi,\ M_{\tilde{h}} \rho_z (I - M_{2r})\eta \rangle| \\
&\le \sum_z |\langle (I - M_{2r})\lambda_f M_r M_{\tilde{k}} \rho_z \xi,\ M_{\tilde{h}} \rho_z (I - M_{2r})\eta \rangle| \\
&\le r^{-1} \|h\|_2 \|k\|_2 J_D(f) \|\xi\|_2 \|\eta\|_2,
\end{aligned}
$$

for reasons given near the end of the proof of Proposition 3.1. □

Corollary 3.7. *For given $r > 0$, suppose that $E \subset B(r)$ and $F \subset G \setminus B(2r)$, and set $k = \chi_E$ and $h = \chi_F$. Then for any $f \in c_c$ we have*

$$\|\lambda_{(h^* * k)f}\| \le r^{-1} |E|^{1/2} |F|^{1/2} J_D(f),$$

and

$$J_D((h^* * k)f) \le |E|^{1/2} |F|^{1/2} J_D(f).$$

3.3. **Cutoff functions approximating indicator functions of annuli.**

Notation 3.8. For $t > s > 0$ we define the annulus $A(s,t)$ to be

$$A(s,t) = B(t) \setminus B(s) = \{x \in G : s < \mathbb{L}(x) \le t\}. \tag{3.2}$$

Corollary 3.9. *For given $t > s > 2r > 0$ let $k = |B(r)|^{-1}\chi_{B(r)}$ and $h = \chi_{A(s,t)}$, and let $g = h^* * k$. Then for any $f \in c_c$ we have*

$$\|\lambda_{gf}\| \le r^{-1} (|B(r)|^{-1} |B(t)|)^{1/2} J_D(f),$$

and

$$J_D(gf) \le (|B(r)|^{-1} |B(t)|)^{1/2} J_D(f).$$

One can consider here that we are interested in restricting f to $A(s,t)$, as $\chi_{A(s,t)} f$, but we are first "smoothing" $\chi_{A(s,t)}$ by convolving it with the probability function k centered at 0, to give gf.

The following facts are easily verified:

Lemma 3.10. *For g defined as in Corollary 3.9, we have $0 \le g \le 1$, and furthermore*

- a) *If $g(x) \ne 0$ then $s - r < \mathbb{L}(x) \le t + r$, that is, $x \in A(s-r, t+r)$.*
- b) *If $x \in A(s+r, t-r)$, that is, $s + r < \mathbb{L}(x) \le t - r$, then $g(x) = 1$.*

For later use we draw the following consequences from Corollary 3.9 and the above lemma. Suppose that $t > s > 2r > 0$, and suppose that $f \in c_c$ vanishes identically on both the annuli $A(s-r, s+r)$ and $A(t-r, t+r)$. Then

$$\|\lambda_{f\chi_{A(s+r,t-r)}}\| \le r^{-1} (|B(r)|^{-1} |B(t)|)^{1/2} J_D(f).$$

and

$$J_D(f\chi_{A(s+r,t-r)}) \le (|B(r)|^{-1} |B(t)|)^{1/2} J_D(f).$$

If we reparametrize this inequality by sending t to $t+r$ and s to $s-r$ we obtain the following result:

Proposition 3.11. *Suppose that* $t > s > 3r > 0$*, and suppose that* $f \in c_c$ *vanishes identically on both the annuli* $A(s-2r, s)$ *and* $A(t, t+2r)$*. Then*

$$\|\lambda_{f\chi_{A(s,t)}}\| \leq r^{-1}(|B(r)|^{-1}|B(t+r)|)^{1/2} J_D(f).$$

and

$$J_D(f\chi_{A(s,t)}) \leq (|B(r)|^{-1}|B(t+r)|)^{1/2} J_D(f).$$

4. Application to Nilpotent-by-finite groups

We assume for the remainder of the paper that $\mathbb{L}$ is a length function with the property of bounded doubling.

Notation 4.1. For a fixed $R \in \mathbb{R}$ with $R \geq 2$, and for any natural numbers m, n, we set $\tilde{B}(n) = B(R^n)$ and we set $\tilde{A}(m,n) = A(R^m, R^n)$. For $n \geq 1$ we then set $k_n = |\tilde{B}(n-1)|^{-1}\chi_{\tilde{B}(n-1)}$ and $h_n = \chi_{\tilde{A}(n,n+1)}$, and $g_n = h_n * k_n$.

These definitions imply that $h_n^* = h_n$, and the support of g_n is contained in $A(R^n - R^{n-1}, R^{n+1} + R^{n-1})$. We now fix a parameter R of the form $R = 2^K$, with $K \in \mathbb{N}$ to be chosen later. In particular, $R \geq 2$. This R will be used implicitly for much of the rest of this section. Then from the inequality (5.1) we obtain

$$|\tilde{B}(n-1)|^{-1}|\tilde{B}(n+1)| \leq C_{\mathbb{L}}^{2K}.$$

Notice that the bound on the right is independent of n. Notice also that $R^{n+1} - R^{n-1} \geq 2R^{n-1}$ because $R \geq 2$.

In the series of results below we employ the following notation. By C_k we denote a finite, positive quantity which depends only on the constant $C_{\mathbb{L}}$ in the formulation (1.2) of the bounded doubling hypothesis for $\mathbb{L}$, and on the supplementary quantity R which is to be chosen later in the proof. In particular, each C_k is independent of quantities n, N that appear in the analysis. Explicit expressions for each of these constants as functions of $C_{\mathbb{L}}, R$ can be extracted from the steps below, but their precise values are of no intrinsic significance for our purposes.

We can apply Corollary 3.9 to obtain:

Lemma 4.2. *For any* $f \in c_c$ *and for any* $n \geq 1$ *we have*

$$\|\lambda_{g_n f}\| \leq C_1 R^{-n} J_D(f)$$

where $C_1 = C_{\mathbb{L}}^K$.

It is natural to ask whether there exist length functions without bounded doubling for which this lemma has an analogue.

Proposition 4.3. *If $|n-m| \geq 2$ then g_n and g_m have disjoint support.*

Proof. We can assume that $n > m$. If $g_m(x) \neq 0$ then $\mathbb{L}(x) \leq R^{m+1} + R^{m-1}$, while if $g_n(x) \neq 0$ then $R^n - R^{n-1} < \mathbb{L}(x)$. But $R^{m+1} + R^{m-1} < R^n - R^{n-1}$ because $R \geq 2$ and $n - m \geq 2$. □

In particular, g_{2n} and $g_{2(n+1)}$ have disjoint support. Because of this, we for the moment restrict to using these functions. From Lemma 4.2 and $R \geq 2$ we obtain, for any integer $N \geq 1$,

$$\| \sum_{n \geq N} \lambda_{g_{2n}f} \| \leq \sum_{n \geq N} R^{-2n} C_1 J_D(f) = 2C_1 R^{-2N} J_D(f).$$

Accordingly:

Notation 4.4. Set $p_N = p_N^f = \sum_{n \geq N} g_{2n} f$.

We then have:

Proposition 4.5. *For any integer $N \geq 1$*

$$\|\lambda_{p_N}\| \leq 2C_1 R^{-2N} J_D(f).$$

Although Proposition 3.2 gives some information about $L_D(g_n f)$, we have not seen how to get a useful bound for $L_D(p_N)$. In contrast, by using the support properties of the g_n's we can obtain the following useful bound for $J_D(p_N)$, that is independent of N:

Proposition 4.6. *For any positive integer N,*

$$J_D(p_N^f) \leq C_2 J_D(f)$$

where $C_2 = 4RC_1$.

Proof. Fix N, and let $r > 0$ be given. Let N_r be the biggest M such that for $n < M$ the annulus $A(R^{2n} - R^{2n-1}, R^{2n+1} + R^{2n-1})$ is contained in $B(r)$, that is, such that $R^{2n+1} + R^{2n-1} \leq r$. If $\xi \in c_c$ has its support in $B(r)$, then for any $n < N_r$ the support of $\lambda_{g_{2n}f}\xi$ is contained in $B(2r)$, and so $(I - M_{2r})\lambda_{g_{2n}f}\xi = 0$. Thus, for $n < N_r$

$$(I - M_{2r})\lambda_{g_{2n}f} M_r = 0.$$

Consequently, by Lemma 4.2

$$\begin{aligned} \|(I - M_{2r})\lambda_{p_N} M_r\| &\leq \sum_{n \geq N_r} \|\lambda_{g_{2n}f}\| \\ &\leq \sum_{n \geq N_r} C_1 R^{-2n} J_D(f) = 2C_1 R^{-2N_r} J_D(f). \end{aligned}$$

Now from the definition of N_r we have

$$r \leq R^{2N_r+1} + R^{2N_r-1} \leq 2R^{2N_r+1}$$

because $R \geq 2$. Thus $R^{2N_r} \geq r/(2R)$. On using this in the previous displayed equation, we obtain:

$$\|(I - M_{2r})\lambda_{p_N} M_r\| \leq 2(2R/r)C_1 J_D(f).$$

Since this is true for all $r > 0$, the proof is complete. □

Now set $q_N = q_N^f = f - p_N$. Notice that $q_N(x) = 0$ when for some $n \geq N$ we have $g_{2n}(x) = 1$, which from Lemma 3.10 happens when

$$R^{2n} + R^{2n-1} < \mathbb{L}(x) \leq R^{2n+1} - R^{2n-1}.$$

Thus q_N is supported in the union of the annular regions $A_n = A(s_n, t_n)$, with

$$s_n = R^{2(n-1)+1} - R^{2(n-1)-1} \text{ and } t_n = R^{2n} + R^{2n-1}.$$

We now arrange to apply Proposition 3.11 to control $\lambda_{f\chi_{A(s_n,t_n)}}$. We seek r_n such that $3r_n < s_n = R^{2n-3}(R^2-1)$. To ensure that q_N^f vanishes on $A(s_n - 2r_n, s_n)$ it suffices to have $s_n - 2r_n \geq R^{2(n-1)} + R^{2(n-1)-1}$, that is,

$$2r_n < R^{2n-1} - R^{2n-2} - 2R^{2n-3} = R^{2n-3}(R^2 - R - 2),$$

while its vanishing on $A(t_n, t_n + 2r_n)$ is ensured if $t_n + 2r_n \leq R^{2n+1} - R^{2n-1}$, that is, if

$$2r_n < R^{2n+1} - R^{2n} - 2R^{2n-1} = R^{2n-1}(R^2 - R - 2).$$

Assuming henceforth that $R \geq 4$, it is easily checked that $r_n = \frac{1}{6}R^{2n-1}$ satisfies all three of these conditions.

We can now apply Proposition 3.11. With the values of r_n, s_n, t_n chosen above,

$$A_n = A(s_n, t_n) = A(R^{2(n-1)+1} - R^{2(n-1)-1}, R^{2n} + R^{2n-1}).$$

Then by inequality (5.1),

$$|B(r_n)|^{-1}|B(t_n + r_n)| \leq C_{\mathbb{L}}^{1+\log_2((t_n+r_n)/r_n)} = C_{\mathbb{L}}^{1+\log_2(6R+7)}.$$

The uniform (with respect to n) boundedness of these ratios is crucial to our analysis and relies on the bounded doubling hypothesis. This uniform boundedness, in combination with Proposition 3.11, gives

$$\|\lambda_{(q_N\chi_{A_n})}\| \leq C_3 R^{-2n} J_D(q_N) \tag{4.1}$$

where C_3 depends only on $C_{\mathbb{L}}, R$.

From Proposition 4.6 we obtain

$$J_D(q_N^f) \leq J_D(f) + J_D(p_N^f) \leq (1 + C_2)J_D(f),$$

which together with inequality (4.1) establishes

Lemma 4.7. *With notation as above, for each* n

$$\|\lambda_{(q_N \chi_{A_n})}\| \leq C_4 R^{-2n} J_D(f)$$

where $C_4 = (1 + C_2)C_3$.

Notation 4.8. Set $\rho_N = \rho_N^f = \sum_{n \geq N} q_N^f \chi_{A_n}$.

Notice that if $\mathbb{L}(x) > R^{2N} + R^{2N-1}$ then $\rho_N^f(x) = q_N^f(x)$, so that $f - (p_N + \rho_N)$ is supported in $B(R^{2N} + R^{2N-1})$. Much as in the proof of Proposition 4.5 we obtain from the last displayed inequality above:

Proposition 4.9. *With notation as above, for any integer* $N \geq 2$,

$$\|\lambda_{\rho_N}\| \leq 2C_4 R^{-2N} J_D(f).$$

But we also need control of $J_D(\rho_N)$:

Proposition 4.10. *With notation as above, for any integer* $N \geq 2$

$$J_D(\rho_N^f) \leq 4C_4 J_D(f).$$

Proof. The proof is very similar to that of Proposition 4.6, but we give the details since the bookkeeping is somewhat different. Fix N, and let $r > 0$ be given. Let N_r be the biggest M such that for $n < M$ the annulus A_n is contained in $B(r)$, that is, such that $R^{2n} + R^{2n-1} \leq r$. If $\xi \in c_c$ has its support in $B(r)$, then for any $n < N_r$ the support of $\lambda_{(q_N \chi_{A_n})}\xi$ is contained in $B(2r)$, and so $(I - M_{2r})\lambda_{(q_N \chi_{A_n})}\xi = 0$. Thus, for $n < N_r$ we have

$$(I - M_{2r})\lambda_{(q_N \chi_{A_n})} M_r = 0.$$

Consequently, by Lemma 4.7 we have

$$\begin{aligned}
\|(I - M_{2r})\lambda_{(q_N \chi_{A_n})} M_r\| &\leq \sum_{n \geq N_r} \|\lambda_{(q_N \chi_{A_n})}\| \\
&\leq \sum_{n \geq N_r} R^{-2n} C_4 J_D(f) = 2C_4 R^{-2N_r} J_D(f).
\end{aligned}$$

Now from the definition of N_r we have

$$r \leq R^{2N_r} + R^{2N_r - 1} \leq 2R^{2N_r}$$

because $R \geq 4$. Thus $R^{2N_r} \geq r/2$. On using this in the previous displayed equation, we obtain:

$$\|(I - M_{2r})\lambda_{(q_N \chi_{A_n})} M_r\| \leq 4C_4 r^{-1} J_D(f).$$

Since this is true for all $r > 0$, this concludes the proof. $\square$

Proposition 1.6, and its extension concerning arbitrary functions for which $[D_{\mathbb{L}}, \lambda_f]$ is bounded, have now been established.

We finally assemble the pieces to conclude the proof of our main theorem. Let $\varepsilon > 0$ be given. We will show that the set

$$B_J = \{\lambda_f : f \in c_c,\ f(e) = 0, \text{ and } J_D(f) \leq 1\}$$

can be covered by a finite number of ε-balls for the operator norm. Since $J_D \leq L_D$, this will imply the same result for L_D in place of J_D above, which verifies the criterion of Proposition 1.5, and so proves the assertion of our main theorem. Note that up to this point we have not shown that B_J is bounded for the operator norm.

Fix $R \geq 4$, and choose $N \geq 2$ such that

$$R^{-2N} \max(C_1, C_4) < \varepsilon/4.$$

From Propositions 4.5 and 4.9 it now follows that if $f \in B_J$ then

$$\max\left(\|\lambda_{p_N^f}\|, \|\lambda_{\rho_N^f}\|\right) < \varepsilon/4$$

so that

$$\|\lambda_{p_N^f + \rho_N^f}\| < \varepsilon/2.$$

Thus

$$\|\lambda_f - \lambda_{f-(p_N+\rho_N)}\| < \varepsilon/2.$$

We need next to know that the set of functions of the form $f - (p_N^f + \rho_N^f)$ with $f \in B_J$ is bounded for the operator norm. To do this we first show that it is bounded for the norm J_D. From Propositions 4.6 and 4.10 it follows that for any $f \in B_J$ we have

$$J_D(f - (p_N^f + \rho_N^f)) \leq J_D(f) + C_2 J_D(f) + 4C_4 J_D(f) \leq 1 + C_2 + 4C_4,$$

giving the desired boundedness for J_D.

Now by construction $f - (p_N^f + \rho_N^f)$ is supported in $B(R^{2N} + R^{2N-1})$. Let

$$V_J^N = \{f \in c_c :\ f(e) = 0, \text{ and } f \text{ is supported in } B(R^{2N} + R^{2N-1})\}.$$

Let

$$B_J^N = \{f \in V_J^N : J_D(f) \leq 1 + C_2 + 4C_4\},$$

and notice that each $f - (p_N^f + \rho_N^f)$ is in B_J^N. Both J_D and the operator norm (via λ) restrict to norms on the vector space V_J^N, and these norms are equivalent because V_J^N is finite-dimensional. Thus B_J^N is bounded for the operator norm. Since we have shown above that every $f \in B_J$ is in the operator-norm $\varepsilon/2$-neighborhood of an element of B_J^N, it follows that B_J is bounded for the operator norm.

Since V_J^N is finite-dimensional, B_J^N can be covered by a finite number of operator-norm $\varepsilon/2$-balls. Consequently, since B_J is contained in the operator-norm $\varepsilon/2$-neighborhood of B_J^N, it follows that B_J can be covered by a finite number of operator-norm ε-balls. Thus B_J is totally bounded for the operator norm. This concludes the proof of Theorem 1.4. □

5. On polynomial growth

Proposition 1.2 states that strong polynomial growth implies the bounded doubling property, which implies polynomial growth, and that these are equivalent for finitely generated groups.

Proof of Proposition 1.2. Suppose that $\mathbb{L}$ has strong polynomial growth. Then, with notation as in Definition 1.1, for any strictly positive r, s we get

$$|B(s)| \leq c^2 s^d r^{-d} |B(r)|,$$

which for $s = 2r$ gives the bounded doubling property. Suppose instead that $\mathbb{L}$ has bounded doubling. Then for any $s \geq 1$ we get $|B(2^k s)| \leq C_{\mathbb{L}}^k |B(s)|$ for each nonnegative integer k. From this we find that if $1 \leq s \leq r$, then

$$|B(r)| \leq C_{\mathbb{L}}^{1+\log_2(r/s)} |B(s)| \tag{5.1}$$

where $\log_2$ denotes the base 2 logarithm. Indeed, let k be the positive integer that satisfies $2^{k-1}s < r \leq 2^k s$. Then $|B(r)| \leq |B(2^k s)| \leq C_{\mathbb{L}}^k |B(s)|$ and $k - 1 \leq \log_2(r/s)$. On setting $s = 1$ and rearranging we see that $\mathbb{L}$ has polynomial growth.

Suppose now that G is finitely generated and that $\mathbb{L}$ is a length function on G. Then for any word-length function $\tilde{\mathbb{L}}$ on G there exists $C < \infty$ such that $\mathbb{L} \leq C^{-1}\tilde{\mathbb{L}}$, that is, the balls $\tilde{B}(r)$ associated to $\tilde{\mathbb{L}}$ satisfy $\tilde{B}(r) \subset B(Cr)$. Thus if $\mathbb{L}$ has polynomial growth, it follows that $\tilde{\mathbb{L}}$ does also. According to a theorem of Gromov [8, 10, 11, 25], this implies that G is nilpotent-by-finite. But the property of strong polynomial growth holds for any word-length function on a finitely generated nilpotent-by-finite group [26, 2, 11]. Thus $\tilde{\mathbb{L}}$ has strong polynomial growth, and so there are constants $\tilde{C}_{\tilde{\mathbb{L}}}$ and $\tilde{d}$ such that

$$\tilde{C}_{\tilde{\mathbb{L}}}^{-1} r^{\tilde{d}} \leq |\tilde{B}(r)| \leq |B(Cr)|$$

for all $r \geq 0$. This implies that $\mathbb{L}$ has strong polynomial growth. □

We conclude by exhibiting simple examples illustrating the inequivalence between these growth properties, for groups that are not finitely

generated. Chapter 9 of [11] also contains an interesting discussion of infinitely generated groups that are of locally polynomial growth.

Example 5.1. The function $\mathbb{L}(x) = \ln(2|x|)$ for all $x \neq 0$ on the group $G = \mathbb{Z}$ is a length function that is not of polynomial growth.

The remaining examples are based on infinite direct sums of finite groups. Let $(G_n)_{n\in\mathbb{N}}$ be an arbitrary sequence of finite groups, with identity elements e_n. Let G be the direct sum of all these groups; G consists of all sequences $x = (x_1, x_2, x_3, \dots)$ with $x_n \in G_n$ for all n and $x_n = e_n$ for all but finitely many indices n. Multiplication is defined componentwise. Let $e = (e_1, e_2, \dots)$ be the identity element of G. Let $1 \leq a_1 < a_2 < a_3 < \dots$ be a strictly increasing sequence of positive real numbers satisfying $\lim_{n\to\infty} a_n = \infty$. Define $\mathbb{L} : G \to [0, \infty)$ by $\mathbb{L}(e) = 0$ and $\mathbb{L}(x) = \max_{n:x_n\neq e_n} a_n$ for all $x \neq e$. Then $\mathbb{L}$ is a proper length function. Moreover, if $r = a_n$ then $|B(r)| = \prod_{m=1}^{n} |G_m|$.

Example 5.2. Let $G_n = \mathbb{Z}/2\mathbb{Z}$, the group with 2 elements. Let $a_k = 2^{k^2}$. Then $|B(2^{K^2})| = 2^K$ for all $K \in \mathbb{N}$ and more generally $|B(r)| \leq e^{C\sqrt{\ln(r)}}$ for all $r \geq 2$, for a certain constant $C < \infty$. Thus the growth rate of $\mathbb{L}$ is slower than polynomial, and so $\mathbb{L}$ can not have strong polynomial growth. But if $r \geq 2$ and if the natural number p is such that $2^{p^2} \leq r < 2^{(p+1)^2}$ so that $|B(r)| = 2^p$, then $2r \leq 2^{(p+1)^2}$ so that $|B(2r)| \leq 2^{p+1}$. Thus $|B(2r)| \leq 2|B(r)|$, so that $\mathbb{L}$ has bounded doubling.

Example 5.3. Now choose (G_n) so that $|G_n| > 1$ for all n and $\lim_{n\to\infty} |G_n| = \infty$. Choose $a_n = \prod_{m=1}^{n} |G_m|$. The balls on the product group G satisfy $|B(a_n)| = \prod_{m=1}^{n} |G_m| = a_n$ for all n, and $|B(r)| < r$ for all other $r > 1$, so $\mathbb{L}$ has polynomial growth. However, for $2 \leq r = a_n$, $\frac{|B(r)|}{|B(r/2)|} \geq \frac{|B(r)|}{|B(r-1)|} = |G_n|$ is not bounded above uniformly in n, and so the doubling property does not hold.

The next example shows that $\mathbb{L}$ can have polynomial growth, yet grow irregularly.

Example 5.4. Let G be as above. Choose any two parameters $1 < \gamma_1 < \gamma_2 < \infty$, and let $1 = N_1 < N_2 < N_3 < \dots$ be a sequence tending to infinity. Set $a_1 = 1$ and for $N_k \leq n < N_{k+1}$ choose $a_{k+1}/a_k = \gamma_1$ if k is odd, and $= \gamma_2$ if k is even. Then $\mathbb{L}$ has polynomial growth. However, $\mathbb{L}$ need not have strong polynomial growth. Indeed, it is plainly possible to arrange, by choosing the sequence (N_k) to increase to infinity sufficiently rapidly, that

$$\limsup_{r\to\infty} \frac{\log|B(r)|}{\log r} = \gamma_1^{-1} \quad \text{while} \quad \liminf_{r\to\infty} \frac{\log|B(r)|}{\log r} = \gamma_2^{-1}.$$

Example 5.5. Let G^0 be a finite non-commutative simple group, and let $\gamma > 1$. Choose $G_n = G^0$ for all n, and $a_n = \gamma^n$. Then $\mathbb{L}$ has polynomial growth, yet G is not nilpotent-by-finite.

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